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LETTER TO THE EDITOR

# Duality and the phases of $Z(N)$ spin systems

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**Abstract.** Using a generalised duality transformation and symmetry considerations, we obtain the phase diagram for $Z(N)$ spin models. Using known properties of the Villain model, we conclude that for $N \geq 4$ there are at least three phases, one of them being soft. For $N$ a prime number, $N>3$, we have only three types of phases, two being characterised by symmetry arguments, whereas the third one is soft and has all powers of the order and disorder parameters vanishing. For $N$ not a prime number, $N>4$, we have, in addition to this soft phase, phases characterised by non-vanishing powers of the order or disorder parameter, with $Z(N')$ symmetries being broken, where $N'$ is a divisor of $N$.

Recently two-dimensional $Z(N)$ spin models have been intensely studied (Fradkin and Susskind 1978, Kogut 1979, Domany and Riedel 1979, Elitzur *et al* 1979, Köberle and Swieca 1979, Balian *et al* 1975, Bellisard 1978, Korthals Altes 1978), firstly because they are interesting non-trivial systems in their own right and secondly because they are related to $Z(N)$ gauge systems (Fradkin and Susskind 1978, Elitzur *et al* 1979, Kogut 1979, Horn *et al* 1979, Creutz *et al* 1979), which are the poor man's version of quantum chromodynamics. In this Letter we want to describe the main features of the phase diagrams of these models.

Since Kramers and Wannier (1941), duality has proved to be a powerful tool for examining the behaviour of systems undergoing phase transitions. In this Letter we will characterise the phases of $Z(N)$ models, applying a slightly more general concept of duality together with the following plausible assumption: criticality is continuous in the models' parameters. Together with a rigorous result due to Fröhlich and Lieb (1979), this may be used to show that for every discrete global symmetry of our model there exists a region in parameter space where it is spontaneously broken. Symmetry considerations will play an important role in gathering additional information (Wegner 1972). We will restrict our attention to nearest-neighbour $Z(N)$ models, the most general interaction being given by †

$$H=-\sum_{\langle \vec{i},\vec{j}\rangle}\Big\{J_1\Big[\cos\Big(\frac{2\pi}{N}(n_{\vec{i}}-n_{\vec{j}})\Big)-1\Big]+J_2\Big[\cos\Big(\frac{4\pi}{N}(n_{\vec{i}}-n_{\vec{j}})\Big)-1\Big]$$
$$+\cdots+J_{\bar{N}}\Big[\cos\Big(\frac{2\pi\bar{N}}{N}(n_{\vec{i}}-n_{\vec{j}})\Big)-1\Big]\Big\}$$

† Interactions of the form $\sin[(2\pi/N)\Delta n_{ij}]$ are not considered because they are not of ferromagnetic character and lead to models with very different properties.

$$= -\sum_{\langle \hat{i},\hat{j}\rangle} \left( \frac{J_1}{2}(S^+(\hat{i})S(\hat{j}) + S(\hat{i})S^+(\hat{j}) - 2) \right. \tag{1a}$$

$$+\frac{J_2}{2}[(S^+(\hat{i})S(\hat{j}))^2 + (S(\hat{i})S^+(\hat{j}))^2 - 2] + \cdots$$

$$\left. +\frac{J_{\bar{N}}}{2}[(S^+(\hat{i})S(\hat{j}))^{\bar{N}} + (S(\hat{i})S^+(\hat{j}))^{\bar{N}} - 2] \right) \tag{1b}$$

where $\langle \hat{i}, \hat{j}\rangle$ indicates a sum over nearest-neighbour sites, $\bar{N}$ is the largest integer smaller than or equal to $N/2$ and at each site we have variables $n(\hat{i})$ or $S(\hat{i})$ related by

$$S(\hat{i}) = \exp[(2\pi \mathrm{i}/N)n(\hat{i})], \qquad n(\hat{i}) = 0, 1, 2, \ldots, N-1, \tag{2}$$

i.e. $S(\hat{i})$ equals the $N$ roots of unity: $S^N(\hat{i}) = 1$.

Following Yoneya (1978), we obtain the transfer matrix $T$ corresponding to the interaction (1) in terms of unitary operators $S(i)$ acting on the state space as

$$S(i)|n(i)\rangle = \exp[(2\pi \mathrm{i}/N)n(i)]|n(i)\rangle, \tag{3}$$

operators $R(i)$ which rotate the spin on site $i$ by an angle $2\pi/N$,

$$R(i)|n(i)\rangle = |n(i)+1\rangle (\mathrm{mod}\ N), \tag{4}$$

and a set of functions $f_\alpha(\{K_\beta\})$, $\alpha$, $\beta = 0, 1, \ldots, \bar{N}$, satisfying

$$\sum_{\alpha=0}^{N-1} R^\alpha \exp\left\{ \sum_{\delta=0}^{\bar{N}} K_\delta \left[ \cos\left(\frac{2\pi\alpha\delta}{N}\right) - 1 \right] \right\} = \exp\left( \sum_{\alpha=0}^{N-1} f_\alpha(\{K_\beta\}) R^\alpha \right) \tag{5}$$

where $K_\alpha = -\beta J_\alpha$. For nearest-neighbour interactions $T$ is actually a function of the product $S(i)S^+(i+1)$, of $R(i)$ and the coupling constants $K_\alpha$, that is

$$T = \tau[S(i)S^+(i+1), R(i), \{K_\alpha\}, f_\alpha(\{K_\beta\})]. \tag{6}$$

We now define dual variables (Kadanoff and Ceva 1971) as follows:

$$\sigma(i) = S(i)S^+(i+1), \tag{7a}$$

$$\rho(i) = \prod_{j<i} R^+(j)R^+(i). \tag{7b}$$

The non-local disorder variable $\rho(i)$ satisfies

$$\rho^+(i)\rho(i+1) = R^+(i+1). \tag{8}$$

With the help of the variables (7) our transfer matrix may be expressed as

$$T = c\tau[\rho(i)\rho(i+1), \sigma(i), 2f_\alpha(\{K_\beta\}), \{K_\alpha/2\}], \tag{9}$$

where the function $\tau$ is the same function as in equation (6) and $c$ is some constant. Thus our system is self-dual in the following sense:

$$Z(\{K_\alpha\}, \{f_\beta\}) = c'Z(\{2f_\alpha\}, \{K_\beta/2\}), \tag{10}$$

where $Z(\{K_\alpha\}, \{f_\beta\})$ is the partition function and $c'$ an irrelevant constant.

Before presenting our results, let us recall a rigorous one due to Fröhlich and Lieb (1979) from which it follows that any one of our $Z(N)$ models possesses at least one critical point at a $\beta$ high enough so that spontaneous magnetisation sets in, i.e. for a given set of coupling constants $J_\alpha$ there exists a value $\beta = \beta_c$ where the partition function is non-analytic. As the coupling constants vary this point traces out the critical surface $\Sigma$.

Since the duality transformation $\{K_\alpha\} \to \{\tilde{K}_\alpha\} = \{2f_\alpha\}$ interchanges order and disorder variables, it is immediately clear that a point $\{K_\alpha\}$ satisfying the following $\bar{N}$ equations,

$$f_\alpha(K_1, \ldots, K_{\bar{N}}) = K_\alpha/2, \qquad \alpha = 1, 2, \ldots, \bar{N}, \tag{11}$$

is a critical point if there is only one phase transition, and thus belongs to $\Sigma$. If there is more than one transition the transformation $\{K_\alpha\} \to \{2f_\alpha\}$ maps one branch of $\Sigma$ onto another one and we will have to resort to symmetry or other considerations in order to pin down the general shape of $\Sigma$. Notice that the duality transformation $\{K_\alpha\} \to \{2f_\alpha\}$ is different from the one introduced by Kramers and Wannier (1941), since it connects different models†. Only when $\{K_\alpha\}$ and $\{2f_\alpha\}$ describe the same model, as is the case for the scalar Potts (1952) or Villain (1975) models, do the two concepts coincide.

Our duality transformation $K_\alpha \to 2f_\alpha$ may be more conveniently expressed in terms of the following variables:

$$x_\alpha \equiv \exp\left\{\sum_{\delta=0}^{N-1} K_\delta\left[\cos\left(\frac{2\pi\alpha\delta}{N}\right) - 1\right]\right\}, \qquad x_{N-\alpha} = x_\alpha. \tag{12}$$

Diagonalising the cyclic matrix $R$ (Wu and Lin 1976), we may immediately solve equation (6) for the functions $f_\alpha$, obtaining the following expression for the dual variables $\tilde{x}_\alpha$‡:

$$\tilde{x}_\alpha \equiv \exp\left\{\sum_{\delta=0}^{N-1} 2f_\delta\left[\cos\left(\frac{2\pi\alpha\delta}{N}\right) - 1\right]\right\} = \sum_{\eta=0}^{N-1} \exp(2\pi i\alpha\eta/N)x_\eta \bigg/ \sum_{\eta=0}^{N-1} x_\eta. \tag{13}$$

This transformation is actually linear, in the plane

$$\sum_{\eta=0}^{N-1} x_\eta = \sqrt{N} \tag{14}$$

which is invariant under the duality transformation $\tilde{x}_\alpha = D(x_\alpha)$. It furthermore admits a set of fixed points lying on a straight line, which may conveniently be obtained as follows.

Recently a rather special type of self-dual (in Kramers and Wannier's sense) model has been introduced by Villain (1975):

$$Z_v = \sum_{\{n_i\}} \exp\left[\sum_{\langle i,j\rangle} g_\beta\left(\frac{2\pi}{N}(n_i - n_j)\right)\right] \tag{15}$$

where

$$\exp g_\beta(x) = \sum_{m=-\infty}^{+\infty} \exp\left(-\frac{\beta}{2}(x - 2\pi m)^2\right). \tag{16}$$

† In statistical mechanics one usually identifies a particular model as one having fixed values of $K_\alpha$, whereas we find it more natural to think in terms of one model only for fixed $N$, possessing variable coupling constants $K_\alpha$.

‡ This duality transformation has recently been used by Domany and Riedel (1979) to study the phase diagram of anisotropic $N$-vector models.

This model is not of the usual variety since its coupling constants $x_\alpha$ depend on the temperature:

$$x_\alpha^{\rm v}(T)=\sum_{m=-\infty}^{+\infty}\exp\left(-\frac{2\pi^2}{N^2T}(\alpha-Nm)^2\right)\Big/\sum_{m=-\infty}^{+\infty}\exp\left(-\frac{2\pi^2m^2}{T}\right). \tag{17}$$

The transformed coupling constants $\tilde{x}_\alpha^{\rm v}(T)$ are given by

$$\tilde{x}_\alpha^{\rm v}(4\pi^2/N^2T)=x_\alpha^{\rm v}(T) \tag{18}$$

and the fixed point is located at (Zamolodchikov 1978)

$$\bar{x}_\alpha^{\rm v}=x_\alpha^{\rm v}(2\pi/N). \tag{19}$$

The self-duality conditions derived from equation (13) yield a series of planes, whose intersection has to contain the self-dual Villain and Potts points. Therefore the solution is a straight line passing through these points,

$$\frac{x-x_1^{\rm P}}{x-\bar{x}_1^{\rm v}}=\frac{x-x_2^{\rm P}}{x-\bar{x}_2^{\rm v}}=\cdots=\frac{x-x_{\bar{N}}^{\rm P}}{x-\bar{x}_{\bar{N}}^{\rm v}}. \tag{20}$$

The Villain model will also be useful in establishing the number of phases of our $Z(N)$ model. For, if one believes José *et al*'s (1977) estimate of the Kosterlitz–Thouless (1973) transition temperature in the XY model, it follows (Elitzur *et al* 1979) that the Villain model has at least three phases for $N>4$.

Instead of spelling out all of our results, we restrict ourselves to describing the state of affairs for some particular $N$'s.

(1) One of the more interesting results is the existence of at least three phases for $N\geqslant4$. In the region $x_1>x_2>\cdots>x_{\bar{N}}$ it is not possible to characterise more than two phases by symmetry considerations only. Yet we know that the Villain model's thermodynamic path passes through this region and this model has more than two phases for $N>4$. Thus there exists an extra phase containing the self-dual line equation (11), which implies that in this phase the duality transformation transforms order into disorder. As we cross from the low-temperature phase into this extra phase, the order parameter goes to zero and consequently in this phase $\langle S\rangle=\langle\rho\rangle=0$. This in turn requires this extra phase to be soft, with power law decaying correlations (Elitzur *et al* 1979). Since the Potts transition for $N>4$ is unique (Hintermann *et al* 1978) and of first order (Baxter 1973), the Potts point cannot touch this phase and the bifurcation of the self-dual line equation (11) has to occur at some other point $E$ on this line between the Potts and the Villain point. At the point $E$ the latent heat should just have vanished. A mean field calculation of the type used by Balian *et al* (1974) shows just these features, as can be seen from figure 1. For $N$ a prime number, all powers of order and disorder variables vanish, i.e.

$$\langle S^n\rangle=\langle\rho^n\rangle=0, \qquad n=1,2,\ldots,\bar{N},$$

and from symmetry considerations the same is expected for any $N$. This soft phase is bordered by two phases with $\langle S\rangle\neq0$, $\langle\rho\rangle=0$ and $\langle S\rangle=0$, $\langle\rho\rangle\neq0$ respectively and exponentially decaying correlations. Since the soft phase is not related to the spontaneous breaking of a discrete symmetry, it survives the $N\to\infty$ limit and shows up in the XY model.

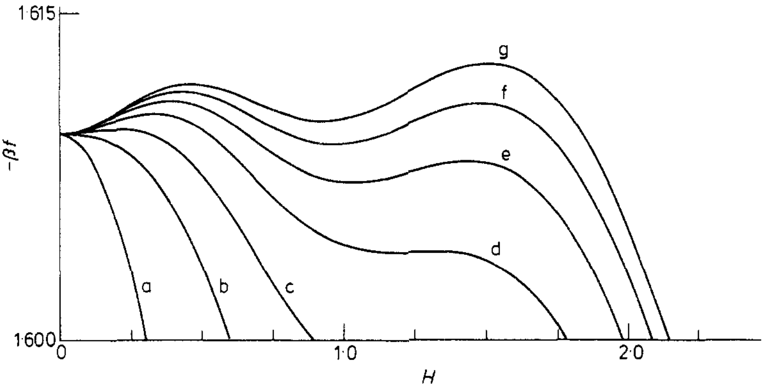


**Figure 1.** Free energy per particle ($f$) versus effective field ($H$) for $N = 5$ with $J_1 = 1$ and $J_2 = 0{\cdot}45$. The curves (a) to (g) correspond to $\beta$ equal to 0·450, 0·495, 0·505, 0·510, 0·513, 0·514 and 0·515 respectively. For each $\beta$ the maximum gives the free energy of the model in the mean field approximation. One transition occurs at $\beta_1 \sim 0{\cdot}500$ and the second one at $\beta_2 \sim 0{\cdot}514$.

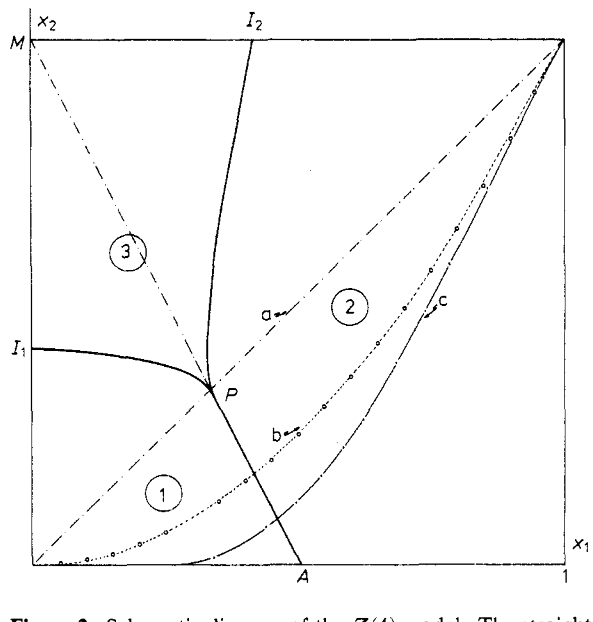


**Figure 2.** Schematic diagram of the $Z(4)$ model. The straight line $AM$ is self-dual. The curves a, b and c represent the thermodynamic paths of the scalar Potts ($J_1 = J_2$), vector Potts ($J_2 = 0$) and Villain model respectively. $P$ is the critical point of the four-state Potts model and $I_1$, $I_2$ are critical Ising points. In phase 1: $\langle S\rangle \neq 0$, $\langle\rho\rangle = 0$, $m \neq 0$; in phase 2: $Z(4)$ inv., $\langle S\rangle = 0$, $\langle\rho\rangle \neq 0$, $m \neq 0$; in phase 3: $Z(2)$ inv., $\langle S\rangle = \langle\rho\rangle = 0$, $\langle S^2\rangle \neq 0$, $\langle\rho^2\rangle \neq 0$, $m = 0$.

(2) For $N = 4$ the phase diagram is already known from Wu and Lin (1974), since it is a special case of the four-state Ashkin–Teller (1943) model. It is shown in figure 2 together with the thermodynamic path of the Villain and $K_2 = K_3 = 0$ model. Duality requires phase 3 to be soft and it is characterised by

$$\langle S\rangle = \langle\rho\rangle = 0, \qquad \text{but } \langle S^2\rangle \neq 0, \langle\rho^2\rangle \neq 0.$$

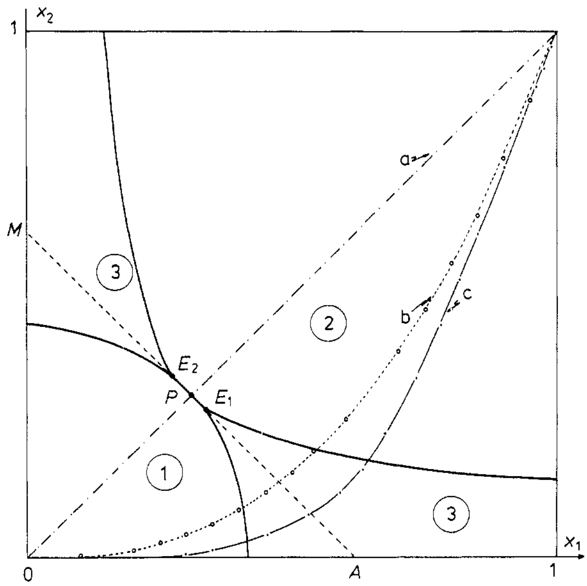


**Figure 3.** Schematic diagram of the $Z(5)$ model. The straight line $AM$ is self-dual. The curves a, b and c represent the thermodynamic paths of the scalar Potts ($J_1 = J_2$), vector Potts ($J_2 = 0$) and Villain model respectively. $P$ is the critical point of the five-state Potts model and $E_1$, $E_2$ are bifurcation points of the self-dual line at which the soft phases originate. In phase 1: $\langle S\rangle \neq 0, \langle \rho\rangle = 0, m \neq 0$; in phase 2: $Z(5)$ inv., $\langle S\rangle = 0, \langle \rho\rangle \neq 0, m \neq 0$; in phase 3: $\langle S^n\rangle = 0, \langle \rho^n\rangle = 0, m = 0$.

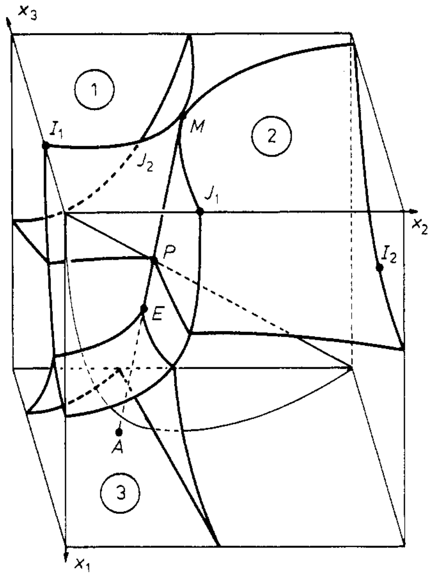


**Figure 4.** Schematic diagram of the $Z(6)$ model. The straight line $AM$ is self-dual. The straight and curved lines going from (0, 0, 0) to (1, 1, 1) are the thermodynamic paths of the six-state Potts and Villain models respectively. The critical points are: $P$ six-state Potts; $I_1$, $I_2$ Ising; $J_1$, $J_2$ three-state Potts. At $E$ the soft phase originates. In phases 1: $\langle S^2\rangle \neq 0$, $\langle \rho^2\rangle = 0$; in phase 2: $\langle S^3\rangle \neq 0$, $\langle \rho^3\rangle = 0$; in phase 3: $\langle S^2\rangle = \langle S^3\rangle = \langle \rho^2\rangle = \langle \rho^3\rangle = 0$, $m = 0$.

(3) For $N=5$ we have spontaneous breaking of the $Z(5)$ symmetry when crossing the straight line, and two soft regions due to the symmetry $Z(K_1, K_2) = Z(K_2, K_1)$ of the partition function, as shown in figure 3.

(4) For $N=6$ our phase diagram is the one of Domany and Riedel (1979), completed by including a soft phase in the region containing the high-temperature fixed point, as shown in figure 4.

(5) For $N=7$ we have again spontaneous breaking of $Z(7)$ at the three straight lines and three soft regions as shown in figure 5. The threefoldness is a consequence of the cyclic symmetry of the partition function in $K_1, K_2, \ldots, K_{\bar{N}}$ valid for any $N$, where $N$ is a prime number.

For ease of presentation we have restricted ourselves to the communication of the above results and defer details to a forthcoming publication, where we will *inter alia* apply the present method straightforwardly to $Z(N)$ gauge systems.

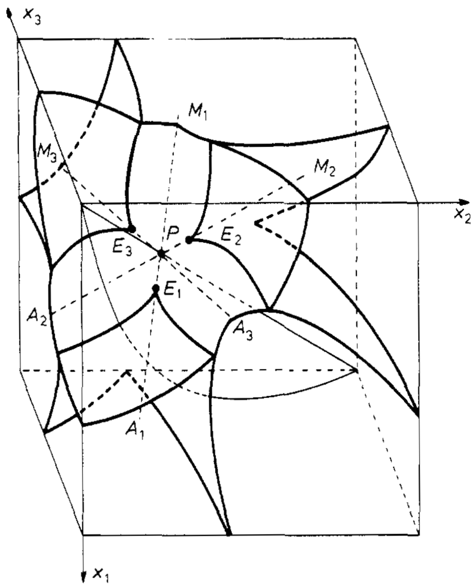


**Figure 5.** Schematic diagram of $Z(7)$ model. The straight lines $A_1M_1$, $A_2M_2$, $A_3M_3$ are self-dual. The straight and curved lines going from $(0, 0, 0)$ to $(1, 1, 1)$ are the thermodynamic paths of the seven-state Potts and Villain models respectively. $P$ is the seven-state Potts critical point and $E_1$, $E_2$, $E_3$ are soft phase bifurcation points.

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